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Explicit evaluation of coupling coefficients for the most degenerate representations of SO(n)

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Abstract. Explicit results for particular coupling coefficients for the most degenerate representations of SO(n) are given. The isoscalar factors which allow a recursive calculation of arbitrary coupling coefficients from those of SO(3) are derived. $6j$- and $9j$-symbols for most degenerate representations are also briefly discussed.

1. Introduction and motivation

Coupling coefficients for the orthogonal groups SO(n) are of great importance and interest in physics. In atomic and nuclear physics these coupling coefficients are used extensively. Another area of application is statistical physics. For example, a high-temperature expansion for the classical n-vector model can be performed to higher orders only if the coupling coefficients for the most degenerate representations of SO(n) are explicitly known. The idea of a group theoretical approach to the evaluation of classical statistical models has been outlined by Joyce [1] for the classical Heisenberg model ($n = 3$). A generalization to arbitrary n is only possible if the corresponding $3j$-symbols are explicitly available. This has been the motivation for us to investigate $3j$-symbols for the most degenerate representations of SO(n) keeping n arbitrary.

There has been substantial progress in the Clebsch–Gordan decomposition of products of irreducible representations for SO(n), SU(n) and Sp($2n$) (see, for example, [2]). However, explicit expressions for the associated Clebsch–Gordan coefficients besides the well-known ones of SU(2) $\simeq$ SO(3) [3, 4] are a rarity. For some explicit results on SU(3) see the excellent text book by Cornwell [2]. Recent progress on SO(n) coupling coefficients, in particular on isoscalars, is due to Ališauskas [5, 6].

In this paper, we calculate $3j$-symbols for the most degenerate representation of SO(n) using the explicit representation functions [7]. In section 2 we recall some basic facts about these representations. Section 3 defines the $3j$-symbols and presents explicit closed form expressions for particular cases. These are the first non-trivial contributions to a high-temperature expansion of the n-vector model mentioned above. After establishing the connection between the $3j$-symbols and Clebsch–Gordan coefficients we present an iteration method for calculating arbitrary $3j$-symbols using

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the isoscalar factors. An explicit expression for these isoscalars is obtained. Finally, in section 6 we briefly discuss integral representations of $6j$- and $9j$-symbols. These integrals appear, for example, in 3-loop and 4-loop contributions to the partition function of n-vector models.

2. The most degenerate representations of SO(n)

The special orthogonal group SO(n) is the set of all linear transformations in the n-dimensional Euclidean space $\mathbb{R}^n$

$$\boldsymbol{x}_i' = g\boldsymbol{x}_i \qquad \boldsymbol{x}_i, \boldsymbol{x}_i' \in \mathbb{R}^n \quad g \in \mathrm{SO}(n) \quad i = 1,2 \tag{1}$$

which preserves the Euclidean norm $|\boldsymbol{x}_i| = |\boldsymbol{x}_i'|$ and the scalar product $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = \boldsymbol{x}_1' \cdot \boldsymbol{x}_2'$. That is, SO($n$) acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$. Here g is an orthogonal $n \times n$ matrix which can be built up by $n(n-1)/2$ simple rotations $g_k(\theta)$ in the planes (x_k, x_{k+1})

$$\begin{pmatrix} x_k' \\ x_{k+1}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}. \tag{2}$$

Any rotation matrix g can be presented in the form $g = g^{(n-1)} \cdots g^{(1)}$ where $g^{(k)} := g_1(\theta_1^k) \cdots g_k(\theta_k^k)$ [7].

A finite-dimensional irreducible representation of SO(n) is uniquely determined by its highest weight [8]

$$[\mu_1, \mu_2, \ldots, \mu_k] \tag{3}$$

with

$$\begin{aligned} &\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{k-1} \geqslant |\mu_k| && \text{for } n = 2k \\ &\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{k-1} \geqslant \mu_k \geqslant 0 && \text{for } n = 2k+1. \end{aligned} \tag{4}$$

The components μ_i are either simultaneously integers (tensorial representations) or half-integers (spinorial representations).

In this paper we consider only unitary irreducible representation of SO(n) in the Hilbert space $H = L^2(S^{n-1})$ of square integrable functions on the unit sphere S^{n-1}. Transformations of SO(n) are defined by left translations

$$\mathcal{D}(g)f(\boldsymbol{x}) := f(g^{-1}\boldsymbol{x}) \qquad g \in \mathrm{SO}(n). \tag{5}$$

$\mathcal{D}(g)$ is called the quasi-regular representation [7].

The Hilbert space H can be decomposed into an orthogonal sum of subspaces H^ℓ of homogeneous polynomials of degree ℓ in n variables. Each invariant subspace H^ℓ carries an irreducible representation of SO(n) with highest weight [8]

$$[\ell, 0, \ldots, 0] \qquad \ell = 0, 1, 2, 3, \ldots. \tag{6}$$

Such a representation which will be denoted by $\mathcal{D}^\ell$ is called the *representation of class one* or *most degenerate representation* of SO(n).

Let us introduce a complete orthonormal basis set $\{|\ell, M\rangle\}$ in H^ℓ where the $(n-2)$-tuple $M := (m_{n-2}, m_{n-3}, \ldots, m_2, m_1)$ enumerates these basis states. The m_i's fulfill the following inequality relations [7, 8]

$$\ell =: m_{n-1} \geqslant m_{n-2} \geqslant \cdots \geqslant m_2 \geqslant |m_1| \qquad m_1 \in \mathbb{Z} \quad m_i \in \mathbb{N}_0 \quad i = 2, \ldots, n-2. \tag{7}$$

The dimension of the space H^ℓ and hence also of the representation $\mathcal{D}^\ell$ is

$$d_\ell := (2\ell + n - 2)\,\frac{(\ell + n - 3)!}{\ell!(n-2)!}. \tag{8}$$

The matrix elements of the representation $\mathcal{D}^\ell$ in the above basis read

$$\mathcal{D}^\ell_{MM'}(g) := \langle \ell, M|\mathcal{D}^\ell(g)|\ell, M'\rangle. \tag{9}$$

Explicitly, the particular matrix elements $\mathcal{D}^\ell_{M0}(g)$, the zero stands for the $(n-2)$-tuple $(0, \ldots, 0)$, are given by a product of Gegenbauer polynomials $C^\nu_k(z)$ [7]

$$\mathcal{D}^\ell_{M0}(g) = \frac{1}{\sqrt{d_\ell}}\, A^{(n)}_{\ell M} \prod_{k=1}^{n-2} \left\{ C^{m_k + k/2}_{m_{k+1} - m_k}(\cos \Phi^{(k+1)}) \sin^{m_k} \Phi^{(k+1)} \right\} \mathrm{e}^{\mathrm{i} m_1 \Phi^{(1)}} \tag{10}$$

where

$$\left[A^{(n)}_{\ell M}\right]^2 := \frac{1}{\Gamma(n/2)} \prod_{k=1}^{n-2} \left\{ \frac{2^{2m_k + k - 2}(m_{k+1} - m_k)!}{\sqrt{\pi}\,\Gamma(m_{k+1} + m_k + k)} (2m_{k+1} + k)\Gamma^2(m_k + k/2) \right\} \tag{11}$$

is the correct normalization factor with respect to the normalized Haar measure $\mathrm{d}g$ on $\mathrm{SO}(n)$

$$\int\limits_{\mathrm{SO}(n)} \mathrm{d}g\, \mathcal{D}^\ell_{M0}(g) \mathcal{D}^{\ell' *}_{M'0}(g) = \frac{\delta_{\ell\ell'}}{d_\ell}\, \delta_{MM'}. \tag{12}$$

In the above $\delta_{MM'}$ stands for the product $\delta_{m_{n-2} m'_{n-2}} \cdots \delta_{m_1 m'_1}$. The angles $\Phi^{(i)}$ in (10) are the polar coordinates of the unit vector $e := (e^1, \ldots, e^n)$ which is the image of the north pole $a := (0, \ldots, 0, 1)$ under the rotation $e = g^{(n-1)} a$

$$\begin{aligned}
e^1 &= \sin \Phi^{(n-1)} \sin \Phi^{(n-2)} \ldots \sin \Phi^{(1)} \\
e^2 &= \sin \Phi^{(n-1)} \sin \Phi^{(n-2)} \ldots \cos \Phi^{(1)} \\
&\vdots \\
e^n &= \cos \Phi^{(n-1)}
\end{aligned} \tag{13}$$

with the conditions $0 \leqslant \Phi^{(1)} \leqslant 2\pi$ and $0 \leqslant \Phi^{(i)} \leqslant \pi$ for $i \neq 1$. Actually, the polar coordinates can be identified with the Euler angles (2) of the rotation matrix

$g^{(n-1)} = g_1(\theta_1^{n-1})\cdots g_{n-1}(\theta_{n-1}^{n-1})$, i.e. $\Phi^{(i)} = \theta_i^{n-1}$. Note that a is invariant under rotations $g^{(k)}$ with $k < n-1$.

The matrix elements $\mathcal{D}^\ell_{M0}(g)$ are invariant under right translations of the subgroup SO$(n-1)$ formed by the set of rotations about the north pole a

$$\mathcal{D}^\ell_{M0}(gh) = \mathcal{D}^\ell_{M0}(g) \qquad \forall h \in \mathrm{SO}(n-1). \tag{14}$$

In other words, SO$(n-1)$ is the stability group of a and contains all rotation matrices of the type $g^{(k)}$ with $k \leqslant n-2$. Matrix elements of the form (14) are called spherical functions and are related to the hyperspherical harmonics in n dimensions by

$$Y_{\ell M}(e) = \sqrt{\frac{d_\ell}{|S^{n-1}|}}\,\mathcal{D}^{\ell\,*}_{M0}(g). \tag{15}$$

Here $|S^{n-1}| := 2\pi^{n/2}/\Gamma(n/2)$ denotes the volume of the unit sphere S^{n-1}. Note that by construction $|e\rangle = \mathcal{D}(g)|a\rangle$. The hyperspherical harmonics are the q-representation of the basis states $|\ell, M\rangle$, i.e. $Y_{\ell M}(e) := \langle e|\ell, M\rangle$. With the relation $\langle a|\ell, M\rangle = \sqrt{d_\ell/|S^n|}\,\delta_{M0}$ one obtains (15). The hyperspherical harmonics form a complete set on S^{n-1} and are orthonormal with respect to the associated Lebesgue measure

$$\int_{S^{n-1}} \mathrm{d}^{n-1}e\, Y_{\ell M}(e) Y^*_{\ell' M'}(e) = \delta_{\ell\ell'}\delta_{MM'}. \tag{16}$$

The measure reads in terms of the polar coordinates (13)

$$\mathrm{d}^{n-1}e = \sin^{n-2}\Phi^{(n-1)}\cdots\sin\Phi^{(2)}\,\mathrm{d}\Phi^{(n-1)}\cdots\mathrm{d}\Phi^{(1)}. \tag{17}$$

For $M = 0$ the spherical functions (14) reduce to the so-called zonal spherical functions

$$\mathcal{D}^\ell_{00}(g) = \frac{\ell!\,(n-3)!}{(\ell+n-3)!}\,C_\ell^{(n-2)/2}(\cos\theta) \tag{18}$$

where $\theta := \Phi^{(n-1)}$ is the polar angle of the unit vector $e = ga$. Or, more generally, if g maps $\boldsymbol{x}$ into $\boldsymbol{x}'$ (cf equation (1)) the angle θ is given by $\cos\theta = \boldsymbol{x}\cdot\boldsymbol{x}'/|\boldsymbol{x}||\boldsymbol{x}'|$.

3. Explicit results for particular $3j$-symbols

In this section we will present explicit expressions for particular $3j$-symbols for the most degenerate representations of SO(n). It has been shown by Girardi *et al* [9] that the Kronecker product of two class-one representations decomposes in a Clebsch–Gordan series as follows ($\ell_1 \geqslant \ell_2$)

$$[\ell_1, 0, \ldots, 0] \otimes [\ell_2, 0, \ldots, 0] = \bigoplus_{l=0}^{\ell_2}\bigoplus_{k=0}^{l} [\ell_1 + \ell_2 - 2l - k, k, 0, \ldots, 0]. \tag{19}$$

We are only interested in the coefficients for the most degenerate representations on the right-hand side, i.e. those with $k = 0$. Let us define the corresponding $3j$-symbols by an integral as follows

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} := \int_{\mathrm{SO}(n)} \mathrm{d}g\, \mathcal{D}^{\ell_1}_{M_1 0}(g) \mathcal{D}^{\ell_2}_{M_2 0}(g) \mathcal{D}^{\ell_3}_{M_3 0}(g). \tag{20}$$

Note that the class-one representations are multiplicity-free. In (20) we have also utilized the fact that the $3j$'s can chosen to be real without loss of generality.

We will first consider the particular case where $M_i = 0$ for all $i = 1,2,3$. In this case (20) reduces to a known integral over three Gegenbauer polynomials

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 = \prod_{i=1}^{3} \left[\frac{\Gamma(n-2)\ell_i!}{\Gamma(\ell_i + n - 2)} \right] \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}$$
$$\times \int_0^\pi \mathrm{d}\theta\, C^{(n-2)/2}_{\ell_1}(\cos\theta)\, C^{(n-2)/2}_{\ell_2}(\cos\theta) C^{(n-2)/2}_{\ell_3}(\cos\theta) \sin^{n-2}\theta. \tag{21}$$

This integral vanishes unless [7] (also see [10] where the special case $n = 3$ is treated)

$$\begin{aligned} &2J := \ell_1 + \ell_2 + \ell_3 \qquad \text{with } J = 0, 1, 2, \ldots \\ &\ell_i = \ell_j + \ell_k, \ell_j + \ell_k - 1, \ldots, |\ell_j - \ell_k|. \end{aligned} \tag{22}$$

Note that these conditions are the same as those known for SO(3). The result of the integration can be given in closed form†

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{\Gamma(J+n-2)}{\Gamma^2(n/2)\Gamma(n-2)\Gamma(J+n/2)}$$
$$\times \prod_{i=1}^{3} \left\{ \frac{\ell_i + (n-2)/2}{d_{\ell_i}} \frac{\Gamma(J - \ell_i + (n-2)/2)}{\Gamma(J - \ell_i + 1)} \right\}. \tag{23}$$

Equation (23) only determines the absolute value for the $3j$-symbols. The relative signs have to be fixed by a phase convention. We adopted the convention

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^J \left| \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \right| \tag{24}$$

which is the same as for $n = 3$ [6]. In table 1 we list some explicit $3j$-symbols of the type (24) for the groups SO(5), SO(6) and SO(7). Note that the $3j$-symbol (24) is invariant under any permutation of ℓ_1, ℓ_2, ℓ_3. Because of this invariance we present in

† It has been mentioned by Vilenkin [7] that the integral appearing in (21) is related to special coupling coefficients. However, no explicit expressions for $3j$-symbols have been given.

Table 1. The square of the $3j$-symbol (23) for the groups SO(5), SO(6) and SO(7). Here, S denotes the sign for the coupling coefficients according to the phase convention (24).

ℓ_1	ℓ_2	ℓ_3	S	SO(5)	SO(6)	SO(7)
0	0	0	+	1	1	1
1	1	0	−	$\frac{1}{5}$	$\frac{1}{2\cdot 3}$	$\frac{1}{7}$
2	1	1	+	$\frac{2}{5\cdot 7}$	$\frac{1}{2^3\cdot 3}$	$\frac{2}{3^2\cdot 7}$
2	2	0	+	$\frac{1}{2\cdot 7}$	$\frac{1}{2^2\cdot 5}$	$\frac{1}{3^3}$
2	2	2	−	$\frac{1}{2\cdot 3\cdot 7}$	$\frac{2}{5^3}$	$\frac{2\cdot 5}{3^4\cdot 11}$
3	2	1	−	$\frac{1}{2\cdot 3\cdot 7}$	$\frac{3}{2^3\cdot 5^2}$	$\frac{1}{3^2\cdot 11}$
3	3	0	−	$\frac{1}{2\cdot 3\cdot 5}$	$\frac{1}{2\cdot 5^2}$	$\frac{1}{7\cdot 11}$
3	3	2	+	$\frac{3^2}{2\cdot 5\cdot 7\cdot 11}$	$\frac{7}{2^3\cdot 5^3}$	$\frac{2^3\cdot 5}{3^2\cdot 7\cdot 11\cdot 13}$
4	2	2	+	$\frac{5}{2\cdot 3\cdot 7\cdot 11}$	$\frac{3}{2^2\cdot 5^3}$	$\frac{2\cdot 7}{3^3\cdot 11\cdot 13}$
4	3	1	+	$\frac{2}{3\cdot 5\cdot 11}$	$\frac{1}{2\cdot 3\cdot 5^2}$	$\frac{2^2}{7\cdot 11\cdot 13}$
4	3	3	−	$\frac{3^2}{2\cdot 5\cdot 11\cdot 13}$	$\frac{3}{5^3\cdot 7}$	$\frac{2}{7\cdot 11\cdot 13}$
4	4	0	+	$\frac{1}{5\cdot 11}$	$\frac{1}{3\cdot 5\cdot 7}$	$\frac{1}{2\cdot 7\cdot 13}$
4	4	2	−	$\frac{2\cdot 7}{3\cdot 5\cdot 11\cdot 13}$	$\frac{2^6}{3\cdot 5^3\cdot 7^2}$	$\frac{2}{7\cdot 11\cdot 13}$
4	4	4	+	$\frac{2^3}{3\cdot 5\cdot 11\cdot 13}$	$\frac{3^4}{5^3\cdot 7^3}$	$\frac{5}{2\cdot 11\cdot 13\cdot 17}$
5	3	2	−	$\frac{5}{2\cdot 3\cdot 11\cdot 13}$	$\frac{1}{2\cdot 5^2\cdot 7}$	$\frac{2}{3^2\cdot 11\cdot 13}$
5	4	1	−	$\frac{1}{11\cdot 13}$	$\frac{1}{2\cdot 3\cdot 7^2}$	$\frac{1}{2\cdot 3\cdot 7\cdot 13}$
5	4	3	+	$\frac{2^3}{3\cdot 5\cdot 11\cdot 13}$	$\frac{3^2}{2^2\cdot 5^2\cdot 7^2}$	$\frac{2\cdot 5^2}{3\cdot 7\cdot 11\cdot 13\cdot 17}$
5	5	0	−	$\frac{1}{7\cdot 13}$	$\frac{1}{2^2\cdot 7^2}$	$\frac{1}{2\cdot 3^3\cdot 7}$
5	5	2	+	$\frac{2^2}{7\cdot 11\cdot 13}$	$\frac{3}{2^5\cdot 7^2}$	$\frac{5^3}{3^4\cdot 7\cdot 13\cdot 17}$
5	5	4	−	$\frac{2^3\cdot 5}{7\cdot 11\cdot 13\cdot 17}$	$\frac{3}{2^3\cdot 7^3}$	$\frac{5^3}{2\cdot 3^4\cdot 13\cdot 17\cdot 19}$

table 1 only those where $\ell_1 \geqslant \ell_2 \geqslant \ell_3$. Vanishing $3j$-symbols have not been included in the table.

For particular combinations of ℓ_1, ℓ_2, ℓ_3 it is also possible to make extensive simplifications for expression (23). Here we mention two examples

$$\begin{pmatrix} \ell & \ell' & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^\ell}{\sqrt{d_\ell}}\,\delta_{\ell,\ell'}$$

$$\begin{pmatrix} \ell & \ell & 2\ell \\ 0 & 0 & 0 \end{pmatrix} = \frac{\Gamma(\ell+n/2)}{d_\ell\,\ell!}\left[\frac{(2\ell)!}{\Gamma(2\ell+n/2)\Gamma(n/2)}\right]^{1/2}$$

which indeed coincide for $n = 3$ with the standard $3j$-symbols of Wigner [6]. These two examples are of particular interest in the high-temperature expansion for the classical n-vector model [11]. They appear as weights of the Θ-topology, which is the first non-trivial contribution to the high-temperature series of the partition function, and have been obtained by Domb [11] only for $\ell = 1, 2, 3$ after lengthy calculations.

Another approach for evaluating $3j$-symbols is to utilize the equivalent definition

$$\mathcal{D}^{\ell_1}_{M_1 0}(g)\mathcal{D}^{\ell_2}_{M_2 0}(g) = \sum_{\ell_3 M_3} d_{\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{D}^{\ell_3 *}_{M_3 0}(g) + \cdots \qquad (25)$$

where the dots indicate the contributions of the terms with $k \geqslant 1$ in (19). Equivalence with the definition (20) becomes obvious by making use of the orthogonality relation (12).

Again let us first consider the particular case for $M_i = 0$, $i = 1,2,3$. The recursion relation of Gegenbauer polynomials [12]

$$(\ell+1)C^{\nu}_{\ell+1}(x) = 2(\ell+\nu)xC^{\nu}_{\ell}(x) - (\ell+2\nu-1)C^{\nu}_{\ell-1}(x)$$

and their relation to the zonal spherical functions (18) lead to the following recursion formula

$$\mathcal{D}^{1}_{00}(g)\mathcal{D}^{\ell}_{00}(g) = \frac{\ell}{2\ell+n-2}\mathcal{D}^{\ell-1}_{00}(g) + \frac{\ell+n-2}{2\ell+n-2}\mathcal{D}^{\ell+1}_{00}(g). \tag{26}$$

A comparison with (25) gives

$$d_{\ell-1}\begin{pmatrix} 1 & \ell & \ell-1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{\ell}{2\ell+n-2} \qquad d_{\ell+1}\begin{pmatrix} 1 & \ell & \ell+1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{\ell+n-2}{2\ell+n-2}.$$

Up to now we have only considered $3j$-symbols with $M = 0$. The simplest one with non-vanishing M's can be obtained from the orthogonality relation (12) with

$$\mathcal{D}^{\ell *}_{M0}(g) = (-1)^{m_1}\mathcal{D}^{\ell}_{\overline{M}0}(g) \tag{27}$$

where

$$\overline{M} := (m_{n-2}, \ldots, m_2, -m_1) \tag{28}$$

is the same as M but with the last component having opposite sign. Because of $\mathcal{D}^{0}_{00}(g) = 1$ we find

$$\begin{pmatrix} \ell & \ell' & 0 \\ M & \overline{M'} & 0 \end{pmatrix} = \frac{(-1)^{\ell-m_1}}{\sqrt{d_\ell}}\delta_{\ell\ell'}\delta_{MM'}. \tag{29}$$

For $n = 3$ this reduces indeed to the known result [6]. The evaluation of more general $3j$-symbols will be the subject of section 5.

4. Connection with Clebsch–Gordan coefficients

Before we consider the evaluation of an arbitrary $3j$-symbol we would like to mention some properties for the Clebsch–Gordan coefficients of class-one representations which are closely related to the $3j$-symbols.

We have seen in the above section that the product space $H^{\ell_1} \otimes H^{\ell_2}$ decomposes into irreducible subspaces as follows

$$H^{\ell_1} \otimes H^{\ell_2} = \bigoplus_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} H^{\ell} \oplus \cdots \tag{30}$$

where only those ℓ values are allowed which fulfill the condition (22). The dots on the right-hand side stand for the terms with non-zero k in the Clebsch–Gordan series (19).

As a basis in this product space we can introduce the states

$$|\ell_1 M_1; \ell_2 M_2\rangle := |\ell_1, M_1\rangle \otimes |\ell_2, M_2\rangle. \tag{31}$$

As an alternative choice to this *product basis* we can choose canonical bases $\{|\ell, M\rangle\}$ in each irreducible subspace H^ℓ. Let us call the corresponding basis a *coupled basis* and denote the basis states for the first orthogonal sum on the right-hand side of (30) by

$$|(\ell_1\ell_2)\ell M\rangle \in \{|\ell, M\rangle \,||\, \ell = |\ell_1 - \ell_2|, \ldots, \ell_1 + \ell_2\}. \tag{32}$$

Note that these states do not form a complete set as the subspaces indicated by dots in (30) also have to be taken into account.

The Clebsch–Gordan coefficients form an unitary matrix which transforms from the product basis (31) to the coupled basis containing (32). We are only interested in the following matrix elements which can be chosen to be real

$$\langle \ell_1 M_1; \ell_2 M_2 | (\ell_1\ell_2)\ell M\rangle = \langle (\ell_1\ell_2)\ell M | \ell_1 M_1; \ell_2 M_2\rangle. \tag{33}$$

The decomposition (30) also implies the following relation for the representation matrices

$$\mathcal{D}^{\ell_1}_{M_10}(g)\mathcal{D}^{\ell_2}_{M_20}(g) = \sum_{\ell_3, M_3} \langle \ell_1 M_1; \ell_2 M_2 | (\ell_1\ell_2)\ell_3 M_3\rangle \langle (\ell_1\ell_2) l_3 0 | \ell_1 0; \ell_2 0\rangle\, \mathcal{D}^{\ell_3}_{M_30}(g) + \cdots. \tag{34}$$

This may be compared with (25) leading to the identification

$$\langle \ell_1 M_1; \ell_2 M_2 | (\ell_1\ell_2)\ell M\rangle = (-1)^{\ell_1 - \ell_2 + m_1}\sqrt{d_\ell} \begin{pmatrix} \ell_1 & \ell_2 & \ell \\ M_1 & M_2 & \overline{M} \end{pmatrix} \tag{35}$$

where again we have adopted a phase convention which for $n = 3$ is that used in standard tables [6]. Here m_1 is the last component of the tuple M.

5. Evaluation of arbitrary 3j-symbols through isoscalars

According to Racah's factorization lemma [13] the Clebsch–Gordan coefficients (CGs) of a group G may be expressed in terms of the CGs of a subgroup H of G. This lemma essentially states that the CGs of G are (ignoring possible multiplicities) proportional to the CGs of its subgroup H. The constant of proportionality is called the *isocalar factor*. A proof of this lemma for the group chains

$$\mathrm{SU}(n) \supset \mathrm{SU}(n-1) \supset \cdots \supset \mathrm{SU}(2)$$

$$\mathrm{SO}(n) \supset \mathrm{SO}(n-1) \supset \cdots \supset \mathrm{SO}(3)$$

$$\mathrm{Sp}(2n) \supset \mathrm{Sp}(2n-2) \supset \cdots \supset \mathrm{Sp}(2)$$

has been given by Klimyk [14].

Here we will utilize Racah's lemma for the calculation of the $3j$-symbols for the most degenerate representation of $SO(n)$ from that of $SO(n-1)$. We will use subscripts or superscripts (n) and $(n-1)$ for any expression which is associated with the group $SO(n)$ and $SO(n-1)$, respectively. The associated isoscalars can be defined by

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ M_1 & M_2 & M_3 \end{pmatrix}_{(n)}$$
$$=: \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^2_{(n)} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 \end{pmatrix}_{(n-1)} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ N_1 & N_2 & N_3 \end{pmatrix}_{(n-1)} . \qquad (36)$$

In the above we assume $n \geqslant 4$ as for $SO(3)$ the $3j$-symbols are well known. Furthermore, the M_i stand, as before, for the $(n-2)$-tuples $M_i = (m^i_{n-2}, \ldots, m^i_1)$ enumerating the basis states for the $SO(n)$ representations. The representations for the subgroup $SO(n-1)$ are labelled by $\lambda_i := m^i_{n-2}$ and for enumeration of the basis we introduced $(n-3)$-tuples $N_i := (m^i_{n-3}, \ldots, m^i_1)$, i.e. $M_i = (\lambda_i, N_i)$. As the $3j$-symbols for $M = N = 0$ are already known explicitly, we may obtain the general $3j$-symbol of $SO(n)$ through (36) from that of $SO(3)$ by induction. However, we still need an explicit expression for the isoscalars. For this we express the left-hand side of (36) in terms of the integral (cf equation (20))

$$\int_{SO(n)} \mathrm{d}g\, \mathcal{D}^{\ell_1(n)}_{M_1 0}(g) \mathcal{D}^{\ell_2(n)}_{M_2 0}(g) \mathcal{D}^{\ell_3(n)}_{M_3 0}(g)$$
$$= \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{S^{n-1}} \mathrm{d}^{n-1}e\; \mathcal{D}^{\ell_1(n)}_{M_1 0}(g) \mathcal{D}^{\ell_2(n)}_{M_2 0}(g) \mathcal{D}^{\ell_3(n)}_{M_3 0}(g) \qquad (37)$$

where we have made use of the invariance (14) of spherical functions. Now using the explicit form (10) we can rewrite $\mathcal{D}^{\ell(n)}_{M0}(g)$ in terms of $\mathcal{D}^{\lambda(n-1)}_{N0}(h)$ where $g = hg_{n-1}(\theta)$

$$\mathcal{D}^{\ell(n)}_{M0}(g) = \frac{A^{(n)}_{\ell M}}{A^{(n-1)}_{\lambda N}} \sqrt{\frac{d^{(n-1)}_\lambda}{d^{(n)}_\ell}}\, C^{\lambda+(n-2)/2}_{\ell-\lambda}(\cos\theta) \sin^\lambda\theta\; \mathcal{D}^{\lambda(n-1)}_{N0}(h). \qquad (38)$$

The normalization factors $A^{(n)}_{\ell M}$ and $A^{(n-1)}_{\lambda N}$ are given in (11). Similarly, we may write for the measure

$$\frac{\Gamma(n/2)}{2\pi^{n/2}} \mathrm{d}^{n-1}e = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \mathrm{d}\theta\, \sin^{n-2}\theta\, \frac{\Gamma((n-1)/2)}{2\pi^{(n-1)/2}} \mathrm{d}^{n-2}e. \qquad (39)$$

Inserting these expressions in (36), the isoscalars can be given by an integral over three Gegenbauer polynomials

$$\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^2_{(n)} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \prod_{i=1}^{3} \left[\frac{A^{(n)}_{\ell_i M_i}}{A^{(n-1)}_{\lambda_i N_i}} \sqrt{\frac{d^{(n-1)}_{\lambda_i}}{d^{(n)}_{\ell_i}}} \right] \int_0^\pi \mathrm{d}\theta\, C^{\lambda_1+(n-2)/2}_{\ell_1-\lambda_1}(\cos\theta)$$
$$\times C^{\lambda_2+(n-2)/2}_{\ell_2-\lambda_2}(\cos\theta)\, C^{\lambda_3+(n-2)/2}_{\ell_3-\lambda_3}(\cos\theta)\, \sin^{\lambda_1+\lambda_2+\lambda_3+n-2}\theta. \qquad (40)$$

Unfortunately, this integral is only known in closed form for $\lambda_1 = \lambda_2 = \lambda_3 = 0$ where it reduces to (21), i.e.

$$\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{bmatrix}^2_{(n)} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2_{(n)} .$$

However, for non-zero λ's we can express the isoscalar as a triple sum.

Let us consider the integral

$$\int_0^\pi d\theta \, \sin^q\theta \, C_{p_1}^{\nu_1}(\cos\theta) \, C_{p_2}^{\nu_2}(\cos\theta) \, C_{p_3}^{\nu_3}(\cos\theta). \tag{41}$$

For its calculation we express the Gegenbauer polynomials as a series in $\cos\theta$ [12]

$$C_p^\nu(\cos\theta) = \frac{1}{\Gamma(\nu)} \sum_{\mu=0}^{[p/2]} \frac{(-1)^\mu \, \Gamma(p+\nu-\mu)}{\mu!\,(p-2\mu)!} \, 2^{p-2\mu} \cos^{p-2\mu}\theta$$

where $[x]$ stands for the integer part of x. Or with the relation $2^{p-2\mu}/\Gamma(p-2\mu+1) = \sqrt{\pi}\,[\Gamma((p+1)/2-\mu)\Gamma((p+2)/2-\mu)]^{-1}$

$$C_p^\nu(\cos\theta) = \frac{\sqrt{\pi}}{\Gamma(\nu)} \sum_{\mu=0}^{[p/2]} \frac{(-1)^\mu \, \Gamma(p+\nu-\mu)}{\Gamma((p+1)/2-\mu)\,\Gamma((p+2)/2-\mu)} \cos^{p-2\mu}\theta. \tag{42}$$

With this, the above integral (41) can be written as

$$\prod_{i=1}^{3} \left\{ \sum_{\mu_i} \frac{\sqrt{\pi}\,(-1)^{\mu_i}\,\Gamma(p_i+\nu_i-\mu_i)}{\Gamma(\nu_i)\,\Gamma(\mu_i+1)\,\Gamma(\frac{p_i+1}{2}-\mu_i)\,\Gamma(\frac{p_i+2}{2}-\mu_i)} \right\}$$
$$\times \int_0^\pi d\theta \, \sin^q\theta \, [\cos\theta]^{p_1+p_2+p_3-2(\mu_1+\mu_2+\mu_3)} . \tag{43}$$

Note that the ranges for the sums are now implicitly defined by the Gamma functions in the denominator. The remaining integral may be performed and leads to a beta function $B(x,y)$

$$\int_0^\pi d\theta \, \sin^q\theta \cos^p\theta = \begin{cases} 0 & \text{for } p \text{ odd} \\ B(\frac{1}{2}(q+1), \frac{1}{2}(p+1)) & \text{for } p \text{ even.} \end{cases}$$

Note that (41) vanishes unless $p_1+p_2+p_3$ is an even integer. For the isoscalar (40) this means that $\sum_{i=1}^3(\ell_i-\lambda_i)$ has to be an even integer.

Inserting everything into (40) the isoscalar factor can be written as a triple sum

$$\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^2_{(n)} = \frac{\Gamma(n/2)\pi}{\Gamma((n-1)/2)} \sum_{\mu_1,\mu_2,\mu_3} B(\Lambda+\tfrac{1}{2}(n-1), J-\Lambda-(\mu_1+\mu_2+\mu_3)+\tfrac{1}{2})$$
$$\times \prod_{i=1}^{3} \left\{ \frac{A_{\ell_i M_i}^{(n)}}{A_{\lambda_i N_i}^{(n-1)}} \sqrt{\frac{d_{\lambda_i}^{(n-1)}}{d_{\ell_i}^{(n)}}} \right.$$
$$\left. \times \frac{(-1)^{\mu_i}\,\Gamma(\ell_i-\mu_i+\frac{n-2}{2})}{\Gamma(\lambda_i+\frac{n-2}{2})\,\Gamma(\mu_i+1)\,\Gamma\left(\frac{\ell_i-\lambda_i+1}{2}-\mu_i\right)\,\Gamma\left(\frac{\ell_i-\lambda_i+2}{2}-\mu_i\right)} \right\} \tag{44}$$

where $2J := \ell_1 + \ell_2 + \ell_3$ and $2\Lambda := \lambda_1 + \lambda_2 + \lambda_3$. This expression is now in a suitable form for numerical evaluation of isoscalars and via (36) also for $3j$-symbols.

More general results, including also the representations with $k > 0$ in (19), are given by Ališauskas [5,6]. Let us point out the interesting observation of Ališauskas [6] that isoscalars of $SO(n)$ are related to coupling coefficients of SO(3). An explanation of this might go as follows. The present derivation is based on the polar coordinate parameterization (13). However, one could also choose, for example, a biharmonic coordinate system where the matrix elements (10) become products of Wigner polynomials, i.e. matrix elements of SO(3) matrices [8]. The latter parameterization seems to be more suitable to shed some light on the observation by Ališauskas mentioned before.

6. Definition and representations of $6j$- and $9j$-symbols

In this section we briefly present various representations for $6j$- and $9j$-symbols. These coupling coefficients show up, for example, in the higher-order terms of a high-temperature expansion for n-vector models [1,11]. The $6j$-symbol appears in the α-graph which is a 3-loop contribution and the $9j$-symbol comes with the A-graph, a 4-loop contribution to the partition function.

The group integral appearing for the α-graph may be used as a definition for the $6j$-symbol as it is a generalization for the integral representation of the $6j$-symbol for SO(3)

$$\int\limits_{SO(n)}\int\limits_{SO(n)}\int\limits_{SO(n)} \mathrm{d}g_1\mathrm{d}g_2\mathrm{d}g_3\, \mathcal{D}^{\ell_1}_{00}(g_1)\mathcal{D}^{\ell_2}_{00}(g_2)\mathcal{D}^{\ell_3}_{00}(g_3)\mathcal{D}^{\ell_4}_{00}(g_2^{-1}g_3)\mathcal{D}^{\ell_5}_{00}(g_3^{-1}g_1)\mathcal{D}^{\ell_6}_{00}(g_1^{-1}g_2)$$
$$=: (-1)^{\ell_4+\ell_5+\ell_6} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_5 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_4 & \ell_2 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} \ell_3 & \ell_4 & \ell_5 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}. \tag{45}$$

From this definition one obtains with (27) and (20) the summation formula

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}$$
$$= \sum_{M_i} (-1)^{\ell_4+\ell_5+\ell_6+m_4+m_5+m_6} \begin{pmatrix} \ell_1 & \ell_5 & \ell_6 \\ 0 & M_5 & \overline{M}_6 \end{pmatrix} \begin{pmatrix} \ell_4 & \ell_2 & \ell_6 \\ \overline{M}_4 & 0 & M_6 \end{pmatrix}$$
$$\times \begin{pmatrix} \ell_3 & \ell_4 & \ell_5 \\ 0 & M_4 & \overline{M}_5 \end{pmatrix}. \tag{46}$$

Here and below m_i stands for the last component of the tuple $M_i = (m^i_{n-2}, \ldots, m^i_1)$, i.e. $m_i := m^i_1$. For $\ell_4 = 0$ follows $M_4 = 0$ and we find, for example

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & \ell_5 & \ell_6 \end{Bmatrix} = \frac{(-1)^{\ell_1+\ell_2+\ell_3}}{\sqrt{d_{\ell_2} d_{\ell_3}}}\, \delta_{\ell_2\ell_5}\delta_{\ell_3\ell_6}. \tag{47}$$

More explicit results on $6j$-symbols of SO(n) are given in [15]. Another summation formula may be obtained from the definition (45) by multiplication with $1 = \int_{\mathrm{SO}(n)} \mathrm{d}g_0$ and the substitution $g_i \to g_0^{-1} g_i$

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix} = \sum_{M_i} (-1)^{\sum_i (\ell_i - m_i)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ \overline{M}_1 & \overline{M}_2 & \overline{M}_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_5 & \ell_6 \\ M_1 & M_5 & \overline{M}_6 \end{pmatrix} \times \begin{pmatrix} \ell_4 & \ell_2 & \ell_6 \\ \overline{M}_4 & M_2 & M_6 \end{pmatrix} \begin{pmatrix} \ell_3 & \ell_4 & \ell_5 \\ M_3 & M_4 & \overline{M}_5 \end{pmatrix}. \tag{48}$$

This is indeed a special case of the general definition for $3nj$-symbols proposed by Derome and Sharp [16] and thus justifies the definition (45).

Similarly, we may define the $9j$-symbol following [16]

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \\ \ell_7 & \ell_8 & \ell_9 \end{Bmatrix} := \sum_{M_i} (-1)^{\sum_i (\ell_i - m_i)} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} \ell_4 & \ell_5 & \ell_6 \\ M_4 & M_5 & M_6 \end{pmatrix} \times \begin{pmatrix} \ell_7 & \ell_8 & \ell_9 \\ M_7 & M_8 & M_9 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ \overline{M}_1 & \overline{M}_2 & \overline{M}_3 \end{pmatrix} \times \begin{pmatrix} \ell_4 & \ell_5 & \ell_6 \\ \overline{M}_4 & \overline{M}_5 & \overline{M}_6 \end{pmatrix} \begin{pmatrix} \ell_7 & \ell_8 & \ell_9 \\ \overline{M}_7 & \overline{M}_8 & \overline{M}_9 \end{pmatrix}. \tag{49}$$

The corresponding integral representation is

$$\int_{\mathrm{SO}(n)} \cdots \int_{\mathrm{SO}(n)} \mathrm{d}g_1 \cdots \mathrm{d}g_6\, \mathcal{D}^{\ell_1}_{00}(g_1^{-1} g_6) \mathcal{D}^{\ell_2}_{00}(g_2^{-1} g_1) \mathcal{D}^{\ell_3}_{00}(g_1^{-1} g_4) \mathcal{D}^{\ell_4}_{00}(g_6^{-1} g_5) \mathcal{D}^{\ell_5}_{00}(g_2^{-1} g_5) \times \mathcal{D}^{\ell_6}_{00}(g_5^{-1} g_4) \mathcal{D}^{\ell_7}_{00}(g_6^{-1} g_3) \mathcal{D}^{\ell_8}_{00}(g_3^{-1} g_2) \mathcal{D}^{\ell_9}_{00}(g_4^{-1} g_3)$$
$$= \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_4 & \ell_5 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_7 & \ell_8 & \ell_9 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \ell_1 & \ell_4 & \ell_7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_5 & \ell_8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_3 & \ell_6 & \ell_9 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \\ \ell_7 & \ell_8 & \ell_9 \end{Bmatrix} \tag{50}$$

and is precisely the integral appearing in the A-graph of a high-temperature expansion of n-vector models.

7. Closing remarks

In this paper we have presented explicit expressions for $3j$-symbols of the most degenerate representations of SO(n). The closed-form expression given in (23) is of particular interest in the calculation of high-temperature properties of n-vector models. They are related to the high-temperature-expansion coefficients defined by Domb [11]

$$c^{(\theta)}_{\ell_1 \ell_2 \ell_3} := d_{\ell_1} d_{\ell_2} d_{\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{51}$$

and thus an explicit expression for arbitrary ℓ_1, ℓ_2, ℓ_3 has been found. Similarly, the higher-order coefficients

$$
\begin{aligned}
c^{(\beta)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6} &:= d_{\ell_1}d_{\ell_2}d_{\ell_3}d_{\ell_4}d_{\ell_5}\begin{pmatrix}\ell_1 & \ell_2 & \ell_5\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix}\ell_3 & \ell_4 & \ell_5\\ 0 & 0 & 0\end{pmatrix}^2\delta_{\ell_5\ell_6}\\
c^{(\gamma)}_{\ell_1\ell_2\ell_3\ell_4\ell_5} &:= d_{\ell_1}d_{\ell_2}d_{\ell_3}d_{\ell_4}d_{\ell_5}\begin{pmatrix}\ell_1 & \ell_2 & \ell_5\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix}\ell_3 & \ell_4 & \ell_5\\ 0 & 0 & 0\end{pmatrix}^2
\end{aligned}
\qquad (52)
$$

are now also available in closed form. Further results will be given elsewhere [17].

It has been mentioned that the motivation to the present study is due to our interest in explicit high-temperature expansions of statistical models with SO(n) symmetry. For this reason we have been working in the polar coordinate parameterization (13) which is most suitable for that aim. However, an analysis similar to the present work can be done using other coordinate systems which will lead to other representations for coupling coefficients of SO(n). In particular, the biharmonic coordinate system used by Barut and Raczka [8] may provide additional insight to the relation between coupling coefficients of SO(n) and those of SO(3) [6, 15].

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References

[1] Jocye G S 1967 *Phys. Rev.* **155** 478
[2] Cornwell J F 1984 *Group Theory in Physics* vol II (London: Academic)
[3] Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (Princeton: Princeton University Press)
Biedenharn L C and van Dam H 1965 *Quantum Theory of Angular Momentum* (New York: Academic)
[4] Rothenberg M, Rivins B, Metropolis N and Wooten J K 1959 *The 3n-j Symbols* (Cambridge, MA: MIT)
[5] Ališauskas S J 1983 *Sov. J. Part. Nucl.* **14** 563
[6] Ališauskas S J 1987 *J. Phys. A: Math. Gen.* **20** 35
[7] Vilenkin N J 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society)
[8] Barut A O and Raczka R 1980 *Theory of Group Representations and Applications* (Warszawa: Polish Scientific)
[9] Girardi G, Sciarrino A and Sorba P 1982 *J. Phys. A: Math. Gen.* **15** 1119; 1982 *Physica* **114A** 365
[10] Gaunt J A 1929 *Phil. Trans. Roy. Soc.* A **228** 151
[11] Domb C 1972 *J. Phys. C: Solid State Phys.* **5** 1417
[12] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* (Berlin: Springer)
[13] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[14] Klimyk A U 1960 *Am. Math. Soc. Transl.* **76** 75
[15] Judd B R, Lister G M S and O'Brien M C M 1986 *J. Phys. A: Math. Gen.* **19** 2473
Judd B R 1987 *J. Phys. A: Math. Gen.* **20** L343
Judd B R, Leavitt R C and Lister G M S 1990 *J. Phys. A: Math. Gen.* **23** 385
[16] Derome J R and Sharp W T 1965 *J. Math. Phys.* **6** 1584
[17] Junker G in preparation